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Comments on new multiple-brane solutions based on Hata-Kojita duality in open string field theory

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Abstract

Recently, Hata and Kojita proposed a new energy formula for a class of solutions in Witten's open string field theory, based on a surprising symmetry of correlation functions they found. Their energy formula can be regarded as a generalization of the conventional energy formula by Murata and Schnabl. Following their proposal, we investigate a new type of double-brane solution, which is connected to our previous solution by the symmetry transformation. Using a regularization method presented in our previous paper, we show that the solution satisfies the equation of motion when contracted to the solution itself and to states in the Fock space. Remarkably, it does not need any phantom piece, which is necessary for the previous one.

Contents

1 Introduction

Since the seminal works of Murata and Schnabl [1,2], solutions for multiple D-branes in Witten's open string field theory [3] are intensively considered. Very recently, there appeared a quite interesting work by Hata and Kojita [4]. They proposed a new way to construct multiple-brane solutions. The purpose of this paper is to examine their new proposal for multiple-brane solutions, which potentially reform our conventional understanding on this subject.

The starting point for the discussion is estimating the energy of the Okawa-type formal solutions

$$\Psi_F = F(K)^2 c \frac{KB}{1 - F(K)^2} c \,, \tag{1.1}$$

where K, B and c are string fields introduced in [5]. These three string fields satisfies simple algebraic relations called KBc subalgebra (see [5,6] for details). The property of the solution Ψ_F is determined by a choice of $F(K)$, which is a function of K. Murata and Schnabl in [1,2] derived a formula which calculates energy of the solution (1.1),

$$E(\Psi_F) = \frac{n_0}{2\pi^2} \,, \tag{1.2}$$

where n_{z_0} denotes the degrees of poles(, or degree of zeros times minus one,) of $G(z) = 1 - F(z)^2$ at $z = z_0$. Here $G(z)$ and $zG(z)^{-1}$ are supposed to be analytic for $\Re(z) \leq 0$ except $z = 0$, and $z = \infty$. So far, the discussion of multiple-brane solutions are based on this formula.

In [4], Hata and Kojita argued that poles at $K = \infty$ are in a sense 'equivalent' to poles at $K = 0$, and it also contribute to the energy. This argument arise from some novel symmetry of the correlation function, as we will see in Section 2. This observation, together with (1.2), leads us to the following

energy formula[*1]:

$$E(\Psi_F) = \frac{1}{2\pi^2}(n_0 + n_\infty)\,. \tag{1.3}$$

The discussion by Hata and Kojita might give a new direction to construct the multiple-brane solutions. However, we need to be careful, since the formula (1.3) seems to conflict with some of existing discussions [7–10]. The poles of $G(z)$ at $z = \infty$ represents singularities related to the identity string fields [10], and we empirically know that we need to be quite careful when we treat this type of singularities. From this perspective, we would like to check the formula (1.3) by a concrete solution and demonstrate how it possesses energy in harmony with (1.3).

In this paper, we consider the following solution:

$$\Psi = Kc\frac{K}{1-K}Bc\,. \tag{1.4}$$

This solution, in the conventional sense, seems to be regular. However, as we will see later, the energy density of the solution is indefinite, and under some proper regularization, it reproduces the energy for double D-branes, as forecast by (1.3). The solution possesses the following properties:

- The solution possesses energy for double D-branes.
- The equation of motion is satisfied when it is contracted to the solution itself and to states in the Fock space.
- However, the Ellwood invariant and the boundary state are for the perturbative vacuum.

The failure for the Ellwood invariant and the boundary state, which is calculated based on our recent paper [9], are common result with the previous double-brane solution [11]. Then, whether these solutions can be regarded as physical objects or not is still open to discussion.

The remainder of this paper is organized as follows: In Section 2, we briefly review a related part of discussion given by Hata and Kojita. In Section 3, we describe our regularization method used in this paper. In Section 4, we calculate energy of solution (1.4) and confirm that the solution satisfies the equation of motion. We also calculate the Ellwood invariant and the boundary state. In Section 5, we summarize our results.

2 Energy counting and symmetry of correlation functions

In this section, we summarize the derivation of the formula (1.3) given by [4]. Our discussion is based on the KBc subalgebra [5]:

$$[K, B] = 0\,, \quad \{B, c\} = 1\,, \quad c^2 = B^2 = 0\,, \tag{2.5}$$

[*1]Note that, we regard (1.3) as a naive expectation, and we do not explicitly write a potential anomalous term in it. To be precise, the discussion of [4] is based on a particular regularization scheme, and the original expression contains anomalous terms in it.

$$QK = 0\,, \quad QB = K\,, \quad Qc = cKc\,. \tag{2.6}$$

This is a quite convenient tool to describe the wedge-based basis used to construct the analytic solution for the tachyon condensation in [12]. Using K, B and c, we can construct the following formal solution:

$$\Psi_{F(K)} = F(K)^2 c \frac{KB}{1-F(K)^2} c\,. \tag{2.7}$$

In [1, 2], Murata and Schnabl proposed the following formula which calculates energy of the solution (2.7):

$$E = \frac{1}{2\pi^2} n_0\,, \tag{2.8}$$

where n_0 denotes the degree of poles of $G(z) = 1 - F(z)^2$ at $z = 0$. Here $G(z)$ and $zG(z)^{-1}$ are supposed to be holomorphic for $\Re(z) \leq 0$ except $z = 0$, and around $z = \infty$. The important point is that energy of the solution are determined by the behavior of the function $G(z)$ around $z = 0$.

Hata and Kojita proposed a more general formula, which is applicable when $G(z)$ possess poles at $z = \infty$. Let us start start from a special case of homomorphisms of the KBc subalgebra [10, 9],

$$\tilde{K} = \frac{1}{K}\,, \quad \tilde{B} = \frac{B}{K^2}\,, \quad \tilde{c} = cK^2Bc\,. \tag{2.9}$$

These tilded string fields, $\tilde{K}$, $\tilde{B}$ and $\tilde{c}$, satisfy the same algebraic relations as the original K, B and c. As proved in [4], the correlation functions in KBc subalgebra are invariant under this transformation:

$$\mathrm{tr}[\tilde{B}\tilde{c}e^{x_1\tilde{K}}\tilde{c}e^{x_2\tilde{K}}\tilde{c}e^{x_3\tilde{K}}\tilde{c}e^{x_4\tilde{K}}] = \mathrm{tr}[Bce^{x_1K}ce^{x_2K}ce^{x_3K}ce^{x_4K}]\,. \tag{2.10}$$

Remarkably, it follows that energy of the solution corresponding to $G(z)$ is the same as that corresponding to $G(1/z)$:

$$E(\Psi_{F(K)}) = E(\Psi_{F(1/K)})\,. \tag{2.11}$$

This relation is suggested by the transformation low of the Okawa-type solution,

$$F(\tilde{K})^2\tilde{c}\frac{\tilde{K}\tilde{B}}{1-F(\tilde{K})^2}\tilde{c} = F(1/K)^2 c\frac{KB}{1-F(1/K)^2}c\,. \tag{2.12}$$

The energy of the solution is calculated, at least naively, only using the form of the solution and the correlation function. Therefore, we expect the relation (2.11) holds. In what follows we call this property (2.11) HK (Hata-Kojita) duality and the transformation (2.9) HK inversion. The equation (2.11) leads to the following energy formula:

$$E = \frac{1}{2\pi^2}(n_0 + n_\infty)\,, \tag{2.13}$$

where n_∞ denotes the degree of poles, or degrees of zeros times minus one, of $G(z)$ at $z = \infty$. Note that, (2.13) is not the same as the expression in [4]. See footnote 1 in pages 1-2. In [4], the authors further developed discussion and considered the case where the function $G(z)$ possess poles on the

negative real axis. We here do not take up these subjects in detail, since, as the purpose of this paper is concerned, they are not necessary for our discussion.

The following points are especially noteworthy: in their discussion, property of formal pure-gauge solutions was discussed using the invariance of the correlation function under the transformation (2.9). It is then likely that the transformation law of the correlation functions under the general homomorphisms of KBc subalgebra determines structures of the formal pure-gauge solutions.

3 Regularization

In this section, we describe the regularization method used in this paper. It is different from that used in [4], which is a variety of ϵ-regularization [13, 2]. Our method is related to the sharp cutoff regularization, and the calculation is straightforward in a sense. The basic idea is the same as that appearing in [11].

Consider a quantity $\varphi(\Lambda)$ (string field) with a large cutoff parameter Λ, and a function of two $\varphi(\Lambda)$'s, $\mathcal{F}(\varphi(\Lambda_1),\, \varphi(\Lambda_2))$. For notational simplicity, we define $f(\Lambda_1,\, \Lambda_2)$ as follows:

$$f(\Lambda_1,\, \Lambda_2) \equiv \mathcal{F}(\varphi(\Lambda_1),\, \varphi(\Lambda_2))\,. \tag{3.14}$$

A typical example is the kinetic term,

$$\mathcal{F}(\varphi(\Lambda_1),\, \varphi(\Lambda_2)) = \mathrm{tr}[\,\varphi(\Lambda_1)\, Q\varphi(\Lambda_2)\,]\,,$$

which is our central interest in the next section. We assume that the function f satisfies the following conditions:

$$(1) \quad f(\Lambda_1,\, \Lambda_2) \text{ is bounded for } 0 \le \Lambda_1,\, \Lambda_2 < \infty\,, \tag{3.15}$$

$$(2) \quad \lim_{\Lambda\to\infty} f(\Lambda,\, a\Lambda) \text{ is continuous as a function of } 0 \le a \le \infty\,. \tag{3.16}$$

Now, we define a regularized quantity φ_R as follows:

$$\varphi_R(\Lambda) \equiv \int_0^1 ds\, \varphi(\lambda(\Lambda,\, s))\,, \tag{3.17}$$

where

$$\lambda(\Lambda;\, s) = (\Lambda+1)^s - 1\,. \tag{3.18}$$

The following property is important for our discussion: for $a > 0$,

$$\lim_{\Lambda\to\infty} \frac{\lambda(a\Lambda;\, s_1)}{\lambda(\Lambda;\, s_2)} = \lim_{\Lambda\to\infty} \frac{(a\Lambda+1)^{s_1}}{(\Lambda+1)^{s_2}} = \begin{cases} \infty & s_1 > s_2\,, \\ 0 & s_2 > s_1\,. \end{cases} \tag{3.19}$$

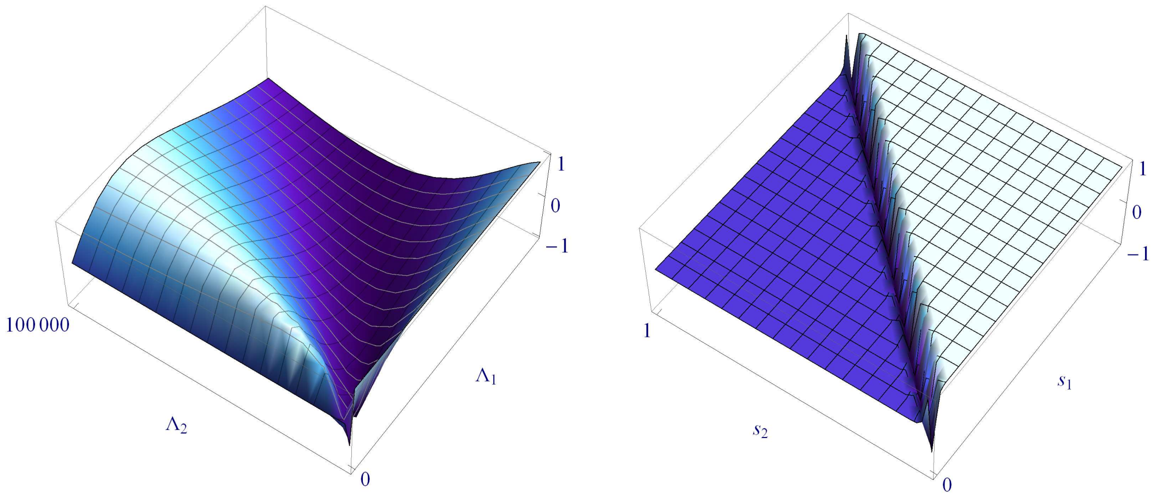

Figure 1: The left graph represents $f(\Lambda_1,\,\Lambda_2)$ in (3.21) for $0 \le \Lambda_1,\,\Lambda_2 \le 10^5$, while the right graph represents $f_0(s_1,\,s_2;\,\Lambda) = f(\lambda(\Lambda;\,s_1),\,\lambda(\Lambda;\,s_2))$ for $\Lambda = 10^{50},\ 0 \le s_1,\,s_2 \le 1$.

Thanks to this property, when Λ is sufficiently large, the function $f_0(s_1,\,s_2;\,\Lambda) = f(\lambda(\Lambda;\,s_1),\,\lambda(\Lambda;\,s_2))$ can be approximated as follows: for $s_1 > s_2$,

$$\begin{aligned} f_0(s_1,\,s_2;\,\Lambda) &\cong f(\Lambda_1,\,\Lambda_2)\Big|_{\Lambda_1 \ll \Lambda_2,\,\Lambda_1 \to \infty} \\ &\cong \lim_{\Lambda_1 \to \infty}\left(\lim_{\Lambda_2 \to \infty} f(\Lambda_1,\,\Lambda_2)\right)\,, \end{aligned}$$

and for $s_2 > s_1$,

$$\begin{aligned} f_0(s_1,\,s_2;\,\Lambda) &\cong f(\Lambda_1,\,\Lambda_2)\Big|_{\Lambda_2 \ll \Lambda_1,\,\Lambda_2 \to \infty} \\ &\cong \lim_{\Lambda_2 \to \infty}\left(\lim_{\Lambda_1 \to \infty} f(\Lambda_1,\,\Lambda_2)\right)\,. \end{aligned}$$

Then, the following identity holds: for $a > 0$,

$$\lim_{\Lambda \to \infty} \mathcal{F}(\varphi_R(a\Lambda),\,\varphi_R(\Lambda)) = \frac{1}{2}\lim_{\Lambda_1 \to \infty}\left(\lim_{\Lambda_2 \to \infty} f(\Lambda_1,\,\Lambda_2)\right) + \frac{1}{2}\lim_{\Lambda_2 \to \infty}\left(\lim_{\Lambda_1 \to \infty} f(\Lambda_1,\,\Lambda_2)\right)\,. \tag{3.20}$$

To illustrate the situation, we present graphs of

$$f(\Lambda_1,\,\Lambda_2) = \sin\left(\frac{7\pi\Lambda_1}{3(\Lambda_1 + \Lambda_2)}\right)\,, \tag{3.21}$$

and corresponding $f_0(s_1,\,s_2;\,\Lambda)$ in Figure 1. Actually, the above relation (3.20) holds under milder conditions, however we shall not pursue this matter here in detail.

Let us apply the discussion above for regularization of string fields with small cutoff parameters. Let us define a quantity $\phi(\epsilon)$ with a small parameter ϵ as follows:

$$\phi(\epsilon) = \varphi(1/\epsilon)\,. \tag{3.22}$$

Then, we can rewrite (3.17) as follows:

$$\begin{aligned} &\int_0^\infty ds\, \varphi(\lambda(\Lambda,\, s)) \\ &= \int_{1/\Lambda}^\infty dx\, \phi(x) \frac{1}{x^2} \left(\frac{d\lambda(s,\, \Lambda)}{ds} \right)^{-1} , \end{aligned} \tag{3.23}$$

where $x = (\lambda(s,\, \Lambda))^{-1}$. Therefore, the above regularization (3.17) is equivalent to the following regularization of ϕ:

$$\phi_R(\epsilon) = \int_0^\infty dx\, \delta_\epsilon(x) \phi(x)\,, \tag{3.24}$$

with the delta sequence

$$\delta_\epsilon(x) = \begin{cases} \dfrac{1}{\log(1/\epsilon + 1)} \dfrac{1}{x(x+1)} & (\epsilon < x)\,, \\ 0 & (0 < x < \epsilon)\,. \end{cases} \tag{3.25}$$

Then, corresponding to (3.20), the following relation holds:

$$\lim_{\epsilon \to 0} \mathcal{F}(\phi_R(\epsilon),\, \phi_R(\epsilon)) = \frac{1}{2} \lim_{x \to 0} \left(\lim_{y \to 0} g(x,\, y) \right) + \frac{1}{2} \lim_{y \to 0} \left(\lim_{x \to 0} g(x,\, y) \right) , \tag{3.26}$$

where $g(x,\, y) = \mathcal{F}(\phi(x),\, \phi(y))$.

4 New double-brane solution

We have treated in the previous paper regularization of the double-brane solution in relation to the sliver-like singularities. We now examine the same question concerning the singularities related to identity state, and it is done while applying the same idea.

According to the energy formula (1.3), the following solution possess energy for double D-branes:

$$\Psi = Kc \frac{K}{1-K} Bc\,. \tag{4.27}$$

The solution (4.27) is the symmetric counterpart of our old solution under the HK inversion $K \to 1/K$,

$$\Psi = \frac{1}{K} c \frac{K^2 B}{K-1} c\,. \tag{4.28}$$

At first look, this solution (4.27) seems regular. However, there does exist an unobtrusive singularity, as essentially explained in [4], and the energy of the solution does depend on choice of regularization methods. This fact seems consistent with the discussion by Erler [10], for the highest level in the dual $\mathcal{L}^-$ level expansion of Ψ is zero. Then, let us here present the precise definition of the solution in question:

$$\begin{aligned} \Psi \equiv &\lim_{\epsilon \to 0} \int_0^\infty dx\, \delta_\epsilon(x) \left(\int_0^\infty du\, e^{-u} \frac{\partial}{\partial x} \frac{\partial}{\partial u} e^{xK} c e^{uK} Bc \right) \\ &\left(\sim \lim_{\epsilon \to 0} \int_0^\infty dx\, \delta_\epsilon(x)\, e^{xK} Kc \frac{KB}{1-K} c \right) . \end{aligned} \tag{4.29}$$

As we will see in Section 4.5, the expression (4.27) possesses quite important ambiguity; If we slightly change the definition of the solution, then the energy of it drastically changes.

4.1 Kinetic term

Let us calculate the normalized kinetic term $\widehat{E}_K$ for the solution (4.29), defined by

$$\widehat{E}_K(\Psi) = \frac{\pi^2}{3}\langle \Psi,\, Q\Psi\rangle\,. \tag{4.30}$$

Note that, if Ψ is a multiple-brane solution, the quantity $\widehat{E}_K+1$ represents the multiplicity of D-branes. Using the correlation function[*2]

$$C_K(x,y;u,v) := \mathrm{tr}\left[e^{xK}ce^{uK}Bc\,Q(e^{yK}ce^{vK}Bc)\right], \tag{4.31}$$

we define the quantity $E_K(\eta,\,\epsilon)$ as follows:

$$E_K(\epsilon,\,\eta) = \int_0^\infty du\int_0^\infty dv e^{-u-v}\left(\frac{\partial}{\partial x}\frac{\partial}{\partial y}\frac{\partial}{\partial u}\frac{\partial}{\partial v}C_K(x,y;u,v)\right)\Bigg|_{x=\epsilon,\,y=\eta}. \tag{4.32}$$

Thanks to the relation (3.26), the regularized kinetic term can be expressed as[*3]

$$\widehat{E}_K = \frac{\pi^2}{3}\frac{1}{2}\left(\lim_{\epsilon\to0}\lim_{\eta\to0}+\lim_{\eta\to0}\lim_{\epsilon\to0}\right)E_K(\epsilon,\,\eta) = \frac{\pi^2}{3}\lim_{\epsilon\to0}\lim_{\eta\to0}E_K(\epsilon,\,\eta)\,. \tag{4.33}$$

By a straightforward calculation, we find

$$\begin{aligned}E_K(\epsilon,\,\eta) &= \int_0^\infty du\int_0^\infty dv e^{-u-v}\left(\frac{\partial}{\partial x}\frac{\partial}{\partial y}\frac{\partial}{\partial u}\frac{\partial}{\partial v}C_K(x,y;u,v)\right)\Bigg|_{x=\epsilon,\,y=\eta}\\
&= \int_0^\infty ds\frac{4e^{-s}}{(s+\eta+\epsilon)^8}\\
&\quad\times\left(c(s,\,\epsilon,\,\eta)\cos\left(\frac{2\pi\epsilon}{s+\epsilon+\eta}\right)+c(s,\,\eta,\,\epsilon)\cos\left(\frac{2\pi\eta}{s+\epsilon+\eta}\right)\right.\\
&\quad-\left((\epsilon+\eta)s^6+2(\epsilon+\eta)^2s^5+(\epsilon+\eta)^3s^4\right)\cos\left(\frac{2\pi(\epsilon+\eta)}{s+\epsilon+\eta}\right)\\
&\quad\left.+\,s(s,\,\epsilon,\,\eta)\sin\left(\frac{2\pi\epsilon}{s+\epsilon+\eta}\right)+s(s,\,\eta,\,\epsilon)\sin\left(\frac{2\pi\eta}{s+\epsilon+\eta}\right)\right),\end{aligned} \tag{4.34}$$

where

$$\begin{aligned}c(s,\,\epsilon,\,\eta) = -\epsilon\Big(&s^6+2(\epsilon+\eta)s^5+(\epsilon^2+5\epsilon\eta-2\eta^2)s^4+2\eta((4+\pi^2)\epsilon^2-4\eta^2)s^3\\
&+\eta(7\epsilon^3+2(5+2\pi^2)\epsilon^2\eta-10\epsilon\eta^2-7\eta^3)s^2\\
&+2\eta(\epsilon^4+5\epsilon^3\eta+\pi^2\epsilon^2\eta^2-5\epsilon\eta^3-\eta^4)s\\
&+3\epsilon(\epsilon-\eta)\eta^2(\epsilon+\eta)^2\Big),\end{aligned} \tag{4.35}$$

[*2] For an explicit form of the correlation function (4.31), see [11] for example.

[*3] To be precise, we have to prove that $E_K(x,\,y)$ satisfies the conditions corresponding to (3.15) and (3.16) before we use the relation (4.33). We check these conditions in appendix A.

$$
\begin{aligned}
s(s,\,\epsilon,\,\eta) = \pi\epsilon^2\Big(&-s^5 - 2(\epsilon-\eta)s^4 - (\epsilon^2+2\epsilon\eta-12\eta^2)s^3 \\
&+ 2\eta(-2\epsilon^2+2\epsilon\eta+7\eta^2)s^2 - \eta^2(\epsilon^2-6\epsilon\eta-5\eta^2)s \\
&+ 2\epsilon\eta^3(\epsilon+\eta)\Big)\,.
\end{aligned}
\tag{4.36}
$$

For finite ϵ, we can change the order of the s-integral and the limit $\eta \to 0$. Then, we obtain that

$$
\begin{aligned}
\lim_{\eta\to 0} E_K(\epsilon,\,\eta) &= -4\pi\epsilon^2 \int_0^\infty ds\, \frac{s^3}{(s+\epsilon)^6} e^{-s} \sin\left(\frac{2\pi s}{s+\epsilon}\right) \\
&= -4\pi\epsilon^2 \sum_{j=0}^\infty \frac{(-1)^j}{(2j+1)!} \int_0^\infty ds\, \frac{s^3}{(s+\epsilon)^6} e^{-s} \left(\frac{2\pi s}{s+\epsilon}\right)^{2j+1}\,.
\end{aligned}
\tag{4.37}
$$

Using the integration formula below,

$$
\int_0^\infty ds\, e^{-s} \frac{s^3}{(s+\epsilon)^6} \frac{s^{2j+1}}{(s+\epsilon)^{2j+1}} \sim \frac{1}{(2j+5)(2j+6)} \frac{1}{\epsilon^2} + \mathcal{O}(\epsilon^{-1})\,,
\tag{4.38}
$$

we can complete the calculation,

$$
\begin{aligned}
\lim_{\epsilon\to 0}\lim_{\eta\to 0} E_K(\epsilon,\,\eta) &= -4\pi\epsilon^2 \sum_{j=0}^\infty \frac{(-1)^j}{(2j+1)!} (2\pi)^{2j+1} \frac{1}{(2j+5)(2j+6)} \frac{1}{\epsilon^2} \\
&= -4\pi \times \left(-\frac{3}{4\pi^3}\right) \\
&= \frac{3}{\pi^2}\,.
\end{aligned}
\tag{4.39}
$$

In other words, we obtain that

$$
\widehat{E}_K = 1\,.
\tag{4.40}
$$

This result shows that the energy of the solution is for double D-branes.

For confirmation, we numerically compute $\widehat{E}_K(\epsilon,\,\eta)$ for finite ϵ and η using the expression (4.34).

$\epsilon,\,\eta$	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}	10^{-8}	10^{-9}
10^{-2}	-1.25208	0.911164	0.957114	0.955959	0.955835	0.955823	0.955822	0.955821
10^{-3}	0.911164	-1.30535	0.938213	0.995497	0.995452	0.995439	0.995438	0.995438
10^{-4}	0.957114	0.938213	-1.31097	0.941009	0.999473	0.999544	0.999542	0.999542
10^{-5}	0.955959	0.995497	0.941009	-1.31154	0.94129	0.999872	0.999954	0.999954
10^{-6}	0.955835	0.995452	0.999473	0.94129	-1.31159	0.941318	0.999912	0.999995

From this table, we see that $\widehat{E}_K(\epsilon,\,\eta)$ is close to one when both ϵ and η/ϵ are sufficiently small.

4.2 Cubic term

Let us move on to the cubic term. Let us define the regularized cubic term $\widehat{E}_C$ as

$$
\widehat{E}_C(\Psi) = -\frac{\pi^2}{3}\langle\Psi,\,\Psi * \Psi\rangle\,.
\tag{4.41}
$$

We also define the quantity $E_C(\epsilon_1,\,\epsilon_2,\,\epsilon_3)$ as follows:

$$
\begin{aligned}
E_C(\epsilon_1,\,\epsilon_2,\,\epsilon_3) = -\int_0^\infty du \int_0^\infty dv \int_0^\infty dw e^{-u-v-w} \\
\left(\frac{\partial}{\partial x}\frac{\partial}{\partial y}\frac{\partial}{\partial z}\frac{\partial}{\partial u}\frac{\partial}{\partial v}\frac{\partial}{\partial w} C_C(x,\,y,\,z;\,u,\,v,\,w)\right)\Bigg|_{x=\epsilon_1,\,y=\epsilon_2,\,z=\epsilon_3} .
\end{aligned}
\tag{4.42}
$$

The discussion presented in Section 3 can naturally be generalized to the case with three valuables, as mentioned in [11]. Then, $\widehat{E}_C$ is given by

$$
\begin{aligned}
\widehat{E}_C(\Psi) =& \frac{\pi^2}{3}\frac{1}{6}\Big(\lim_{\epsilon_1\to 0}\lim_{\epsilon_2\to 0}\lim_{\epsilon_3\to 0} + \lim_{\epsilon_2\to 0}\lim_{\epsilon_3\to 0}\lim_{\epsilon_1\to 0} + \lim_{\epsilon_3\to 0}\lim_{\epsilon_1\to 0}\lim_{\epsilon_2\to 0} \\
&+ \lim_{\epsilon_1\to 0}\lim_{\epsilon_3\to 0}\lim_{\epsilon_2\to 0} + \lim_{\epsilon_2\to 0}\lim_{\epsilon_1\to 0}\lim_{\epsilon_3\to 0} + \lim_{\epsilon_3\to 0}\lim_{\epsilon_1\to 0}\lim_{\epsilon_2\to 0}\Big) E_C(\epsilon_1,\,\epsilon_2,\,\epsilon_3) \\
=& \frac{\pi^2}{3}\frac{1}{2}\Big(\lim_{\epsilon_1\to 0}\lim_{\epsilon_3\to 0}\lim_{\epsilon_2\to 0} + \lim_{\epsilon_1\to 0}\lim_{\epsilon_3\to 0}\lim_{\epsilon_2\to 0}\Big) E_C(\epsilon_1,\,\epsilon_2,\,\epsilon_3) .
\end{aligned}
\tag{4.43}
$$

By a straightforward calculation, we derive the following expression:

$$
\begin{aligned}
& E_C(\epsilon_1,\,\epsilon_2,\,\epsilon_3) \\
&= \int_0^\infty ds \frac{e^{-s}}{(s+\epsilon_1+\epsilon_2+\epsilon_3)^9} \\
&\Bigg(c_1(s,\,\epsilon_1,\,\epsilon_2,\,\epsilon_3)\cos\left(\frac{2\pi\epsilon_1}{s+\epsilon_1+\epsilon_2+\epsilon_3}\right) + c_1(s,\,\epsilon_2,\,\epsilon_3,\,\epsilon_1)\cos\left(\frac{2\pi\epsilon_2}{s+\epsilon_1+\epsilon_2+\epsilon_3}\right) \\
&+ c_1(s,\,\epsilon_3,\,\epsilon_1,\,\epsilon_2)\cos\left(\frac{2\pi\epsilon_3}{s+\epsilon_1+\epsilon_2+\epsilon_3}\right) + c_2(s,\,\epsilon_1,\,\epsilon_2,\,\epsilon_3)\cos\left(\frac{2\pi(\epsilon_1+\epsilon_2)}{s+\epsilon_1+\epsilon_2+\epsilon_3}\right) \\
&+ c_2(s,\,\epsilon_2,\,\epsilon_3,\,\epsilon_1)\cos\left(\frac{2\pi(\epsilon_2+\epsilon_3)}{s+\epsilon_1+\epsilon_2+\epsilon_3}\right) + c_2(s,\,\epsilon_3,\,\epsilon_1,\,\epsilon_2)\cos\left(\frac{2\pi(\epsilon_3+\epsilon_1)}{s+\epsilon_1+\epsilon_2+\epsilon_3}\right) \\
&+ c_3(s,\,\epsilon_1,\,\epsilon_2,\,\epsilon_3)\cos\left(\frac{2\pi(\epsilon_1+\epsilon_2+\epsilon_3)}{s+\epsilon_1+\epsilon_2+\epsilon_3}\right) + s_1(s,\,\epsilon_1,\,\epsilon_2,\,\epsilon_3)\sin\left(\frac{2\pi\epsilon_1}{s+\epsilon_1+\epsilon_2+\epsilon_3}\right) \\
&+ s_1(s,\,\epsilon_2,\,\epsilon_3,\,\epsilon_1)\sin\left(\frac{2\pi\epsilon_2}{s+\epsilon_1+\epsilon_2+\epsilon_3}\right) + s_1(s,\,\epsilon_3,\,\epsilon_1,\,\epsilon_2)\sin\left(\frac{2\pi\epsilon_3}{s+\epsilon_1+\epsilon_2+\epsilon_3}\right) \\
&+ s_2(s,\,\epsilon_1,\,\epsilon_2,\,\epsilon_3)\sin\left(\frac{2\pi(\epsilon_1+\epsilon_2)}{s+\epsilon_1+\epsilon_2+\epsilon_3}\right) + s_2(s,\,\epsilon_2,\,\epsilon_3,\,\epsilon_1)\sin\left(\frac{2\pi(\epsilon_2+\epsilon_3)}{s+\epsilon_1+\epsilon_2+\epsilon_3}\right) \\
&+ s_2(s,\,\epsilon_3,\,\epsilon_1,\,\epsilon_2)\sin\left(\frac{2\pi(\epsilon_3+\epsilon_1)}{s+\epsilon_1+\epsilon_2+\epsilon_3}\right) + s_3(s,\,\epsilon_1,\,\epsilon_2,\,\epsilon_3)\sin\left(\frac{2\pi(\epsilon_1+\epsilon_2+\epsilon_3)}{s+\epsilon_1+\epsilon_2+\epsilon_3}\right)\Bigg) ,
\end{aligned}
\tag{4.44}
$$

where

$$
\begin{aligned}
&c_1(s,\, x,\, y,\, z)\\
=&4x^2\big(3s^6+6s^5x+9s^5y+9s^5z+3s^4x^2+17s^4xy+17s^4xz+6s^4y^2+12s^4yz+6s^4z^2\\
&+2\pi^2s^3x^2y+15s^3x^2y+2\pi^2s^3x^2z+15s^3x^2z+12s^3xy^2+24s^3xyz+12s^3xz^2-6s^3y^3\\
&-18s^3y^2z-18s^3yz^2-6s^3z^3+9s^2x^3y+9s^2x^3z+4\pi^2s^2x^2y^2+18s^2x^2y^2+8\pi^2s^2x^2yz\\
&+36s^2x^2yz+4\pi^2s^2x^2z^2+18s^2x^2z^2-6s^2xy^3-18s^2xy^2z-18s^2xyz^2-6s^2xz^3-9s^2y^4\\
&-36s^2y^3z-54s^2y^2z^2-36s^2yz^3-9s^2z^4+2sx^4y+2sx^4z+12sx^3y^2+24sx^3yz+12sx^3z^2\\
&+2\pi^2sx^2y^3+3sx^2y^3+6\pi^2sx^2y^2z+9sx^2y^2z+6\pi^2sx^2yz^2+9sx^2yz^2+2\pi^2sx^2z^3+3sx^2z^3\\
&-10sxy^4-40sxy^3z-60sxy^2z^2-40sxyz^3-10sxz^4-3sy^5-15sy^4z-30sy^3z^2\\
&-30sy^2z^3-15syz^4-3sz^5+3x^4y^2+6x^4yz+3x^4z^2+3x^3y^3+9x^3y^2z+9x^3yz^2+3x^3z^3\\
&-3x^2y^4-12x^2y^3z-18x^2y^2z^2-12x^2yz^3-3x^2z^4-3xy^5-15xy^4z-30xy^3z^2-30xy^2z^3\\
&-15xyz^4-3xz^5\big)\,,
\end{aligned}
\tag{4.45}
$$

$$
\begin{aligned}
&c_2(s,\, x,\, y,\, z)\\
=&4(s+z)(x+y)\big(s^6+2s^5z-3s^4x^2-6s^4xy-3s^4y^2-2s^4z^2-2s^3x^3-6s^3x^2y+2\pi^2s^3x^2z\\
&-6s^3xy^2+4\pi^2s^3xyz-3s^3xz^2-2s^3y^3+2\pi^2s^3y^2z-3s^3yz^2-8s^3z^3+2s^2x^3z+6s^2x^2yz\\
&+4\pi^2s^2x^2z^2+6s^2x^2z^2+6s^2xy^2z+8\pi^2s^2xyz^2+12s^2xyz^2-9s^2xz^3+2s^2y^3z+4\pi^2s^2y^2z^2\\
&+6s^2y^2z^2-9s^2yz^3-7s^2z^4+7sx^3z^2+21sx^2yz^2+2\pi^2sx^2z^3+21sxy^2z^2+4\pi^2sxyz^3\\
&-9sxz^4+7sy^3z^2+2\pi^2sy^2z^3-9syz^4-2sz^5+3x^4z^2+12x^3yz^2+3x^3z^3+18x^2y^2z^2\\
&+9x^2yz^3-3x^2z^4+12xy^3z^2+9xy^2z^3-6xyz^4-3xz^5+3y^4z^2+3y^3z^3-3y^2z^4-3yz^5\big)\,,
\end{aligned}
\tag{4.46}
$$

$$
c_3(s,\, x,\, y,\, z)=-4(s+x+y+z)^2s^4(2s-x-y-z)(x+y+z)\,,
\tag{4.47}
$$

$$
\begin{aligned}
&s_1(s,\, x,\, y,\, z)\\
=&-4\pi x^3(s+y+z)(-s-x-y-z)\big(2s^3+s^2x-4s^2y-4s^2z+3sxy+3sxz-6sy^2\\
&-12syz-6sz^2-2xy^2-4xyz-2xz^2\big)\,,
\end{aligned}
\tag{4.48}
$$

$$
\begin{aligned}
&s_2(s,\, x,\, y,\, z)\\
=&4\pi e^{-s}(s+y)^2(x+z)^2(-s-x-y-z)\left(s^3-4s^2y+4sxy-5sy^2+4syz-2xy^2-2y^2z\right)\,,
\end{aligned}
\tag{4.49}
$$

$$
s_3(s,\, x,\, y,\, z)=4\pi s^5(x+y+z)^2(s+x+y+z)\,.
\tag{4.50}
$$

As far as we keep ϵ_1 finite, we can take limits $\epsilon_2 \to 0$ and $\epsilon_3 \to 0$ before we perform the s integral:

$$\lim_{\epsilon_2 \to 0} \lim_{\epsilon_3 \to 0} E_C(\epsilon_1,\, \epsilon_2,\, \epsilon_3) = -4\pi\epsilon_1^2 \int_0^\infty ds\, \frac{s^3}{(s+\epsilon_1)^6} \sin\left(\frac{2\pi\epsilon_1}{s+\epsilon_1}\right) . \tag{4.51}$$

This integral is the same as that appeared in (4.37). Accordingly, we obtain

$$\widehat{E}_C = 1. \tag{4.52}$$

4.3 Equation of motion

So far we confirmed the solution reproduces energy for double D-branes. From these calculations, we also conclude that the equation of motion is satisfied when it is contracted with the solution itself;

$$\mathrm{tr}[\,\Psi\, Q\Psi\,] = -\mathrm{tr}[\,\Psi\,\Psi\,\Psi\,] \quad \left(= \frac{3}{\pi^2}\right) . \tag{4.53}$$

Now, we investigate the equation of motion contracted with states in the Fock space. It is apparently satisfied, for states in the Fock space always can be written as a wedge states of width one with local insertions.

Let ϕ be a state in the Fock space. Then each term of the equation of motion can be written as follows:

$$\begin{aligned}
\mathrm{tr}[\,\Psi\Psi\,\phi\,] =& \mathrm{tr}[\,Kc\frac{KB}{1-K}cKc\frac{KB}{1-K}c\,\phi\,] \\
=& \int_0^\infty du \int_0^\infty dv\, e^{-u-v} \frac{\partial}{\partial x}\frac{\partial}{\partial u}\frac{\partial}{\partial y}\frac{\partial}{\partial v} \mathrm{tr}[\,e^{xK}ce^{uK}Bce^{yK}ce^{vK}Bc\,\phi\,] \\
=& \int_0^\infty du \int_0^\infty dv\, e^{-u-v} \frac{\partial}{\partial x}\frac{\partial}{\partial u}\frac{\partial}{\partial y}\frac{\partial}{\partial v} \\
& \big(\mathrm{tr}[\,e^{xK}ce^{uK}e^{yK}ce^{vK}Bc\,\phi\,] - \mathrm{tr}[\,e^{xK}ce^{uK}ce^{yK}e^{vK}Bc\,\phi\,]\big) \\
=& \int_0^\infty du \int_0^\infty dv\, e^{-u-v} \left(C_\phi^{(1,\,2,\,1)}(0,u,v) - C_\phi^{(1,\,1,\,2)}(0,u,v)\right) ,
\end{aligned} \tag{4.54}$$

$$\begin{aligned}
\mathrm{tr}[\,(Q\Psi)\,\phi\,] =& \mathrm{tr}[\,(QKc\frac{KB}{1-K}c)\,\phi\,] \\
=& \int_0^\infty du\, e^{-u} \frac{\partial}{\partial x}\frac{\partial}{\partial u} \mathrm{tr}[\,(Qe^{xK}ce^{uK}Bc)\,\phi\,] \\
=& \int_0^\infty du\, e^{-u} \frac{\partial}{\partial x}\frac{\partial}{\partial u} \big(\mathrm{tr}[\,e^{xK}cKce^{uK}Bc\,\phi\,] - \mathrm{tr}[\,e^{xK}ce^{uK}cKBc\,\phi\,]\big) \\
=& \int_0^\infty du\, e^{-u} \left(C_\phi^{(1,\,1,\,1)}(0,\,0,\,u) - C_\phi^{(1,\,1,\,1)}(0,\,u,\,0)\right) ,
\end{aligned} \tag{4.55}$$

where we have defined

$$C_\phi(x,\,y,\,z) := \mathrm{tr}[\,e^{xK}ce^{yK}ce^{zK}Bc\,\phi\,] , \tag{4.56}$$

and

$$C_\phi^{(i,j,k)}(x_0,\, y_0,\, z_0) = \frac{\partial^i}{\partial x^i}\frac{\partial^j}{\partial y^j}\frac{\partial^k}{\partial z^k}C_\phi(x,\, y,\, z)\Big|_{x=x_0,\, y=y_0,\, z=z_0} . \tag{4.57}$$

Above expressions are valid as far as $C_\phi(x,\, y,\, z)$ is analytic around $(x,\, y,\, z) = (0,\, u,\, v)$, $(0,\, 0,\, u)$ and $(0,\, u,\, 0)$. Since $C_\phi(x,\, y,\, z)$ is a correlation function of three local insertions with a line integral of b-ghost, $C_\phi(x,\, y,\, z)$ is regular for $0 \le x+y+z < \infty$. Then, using the partial integration,

$$\begin{aligned}
&\int_0^\infty du\, e^{-u} C_\phi^{(1,2,1)}(0,\, u,\, v)\\
&= \Big[\, e^{-u} C_\phi^{(1,1,1)}(0,\, u,\, v)\,\Big]_{u=0}^{u=\infty} + \int_0^\infty du\, e^{-u} C_\phi^{(1,1,1)}(0,\, u,\, v)\\
&= -\, C_\phi^{(1,1,1)}(0,\, 0,\, v) + \int_0^\infty du\, e^{-u} C_\phi^{(1,1,1)}(0,\, u,\, v)\,,
\end{aligned} \tag{4.58}$$

and

$$\begin{aligned}
&\int_0^\infty dv\, e^{-v} C_\phi^{(1,1,2)}(0,\, u,\, v)\\
&= \Big[\, e^{-v} C_\phi^{(1,1,1)}(0,\, u,\, v)\,\Big]_{v=0}^{v=\infty} + \int_0^\infty dv\, e^{-v} C_\phi^{(1,1,1)}(0,\, u,\, v)\\
&= -\, C_\phi^{(1,1,1)}(0,\, u,\, 0) + \int_0^\infty dv\, e^{-v} C_\phi^{(1,1,1)}(0,\, u,\, v)\,,
\end{aligned} \tag{4.59}$$

we conclude that

$$\mathrm{tr}[\,(Q\Psi)\,\phi\,] + \mathrm{tr}[\,\Psi\Psi\,\phi\,] = 0. \tag{4.60}$$

4.4 Boundary state and the Ellwood invariant

The Ellwood invariant of the solution (4.29) is for the perturbative vacuum. Indeed,

$$\mathcal{W}(\Psi) = \lim_{\epsilon\to 0}\lim_{z\to 0}\frac{dF_\epsilon^2(z)}{dz}H_\epsilon(z) = 0\,, \tag{4.61}$$

where

$$F_\epsilon(K) = \int_0^\infty dx\,\delta_\epsilon(x)\,e^{xK}K\,, \qquad H_\epsilon(K) = \frac{K}{1-K}\,. \tag{4.62}$$

Here we used the formula given by Murata and Schnabl [1, 2]. In addition, according the analysis of [8, 9], the boundary state of (4.29) is also for the perturbative vacuum.

Remember that the Ellwood invariant of the solution (4.28) is also for the perturbative vacuum. This correspondence might be interesting, but it seems not to be the consequence of the HK duality since conventional formulae for the Ellwood invariant and boundary states are not invariant under the HK inversion, $K \to 1/K$.

4.5 Remarks on ambiguity of solutions in KBc subalgebra

Now, let us slightly modify the definition of the solution (4.29), and consider the following solution:

$$\Psi = \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} du\, e^{-u} \left(\lim_{x \to 0} \frac{\partial}{\partial x} \frac{\partial}{\partial u} e^{xK} c e^{uK} Bc \right) . \tag{4.63}$$

It is easy to calculate the energy or Ellwood invariant of Ψ. We summarize the property of this solution as follows:

- The solution possesses energy for the perturbative vacuum.
- The equation of motion is satisfied when it is contracted to the solution itself and to states in the Fock space.
- The Ellwood invariant is for the perturbative vacuum.

Then, it follows that the expression (1.4) has ambiguity; at least naively, both (4.63) and (4.29) can be written as the form (1.4). To define the solution, we need to regularize the solution and designate the order of limiting operations. Note that these two solutions (4.63) and (4.29) possess the same component in each level.

5 Summary

In this paper, we investigate the solution (1.4) and demonstrated that the energy formula (1.3) holds in this case. The solution seems regular, in a sense that it possess finite energy density, which corresponds to energy density of double D-branes, and satisfy the equation of motion. However, the gauge-invariant observable and the boundary state are for the perturbative vacuum. These inharmonious results make the physical interpretation of the solution difficult. Further research will be needed before the solution is fully accepted or rejected. In particular, we need to clarify the relation of our results and the discussion by Baba and Ishibashi [14], where the authors proved the correspondence between energy and the gauge-invariant observables in part. It is also important to calculate the boundary state based on the newly-proposed method by Kudrna, Maccaferri and Schnabl [15].

Let us here state one more question about the solution. The solution can be written as a pure-gauge form as follows:

$$\Psi = U^{-1} Q U \,, \tag{5.64}$$

where

$$U = \left(1 + \frac{K}{1-K} Bc \right) \,, \tag{5.65}$$

$$U^{-1} = \left(1 + \frac{K}{1-K} Bc \right)^{-1} = 1 - K + KBc \,. \tag{5.66}$$

If we define the solution as (4.29), we expect that the solution is not true pure gauge. So, the gauge parameter U or U^{-1} must be singular in some sense. Then we need to understand in what sense it is singular and characterize the singularity.

Without doubt, more research is needed on the proposal by Hata and Kojita. While our result are apparently consistent with [4], there exist subtlety related to ordering of limits. For example, if we change the definition of the solution as (4.63), then the energy of the solution changes and the formula (1.3) no longer holds. As implied in [4], it seems that the formula (1.3) only holds under appropriate regularizations. The formula (1.3), at least superficially, contradict some existing works. It is necessary to gain better understanding of the relations between them.

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A Some limits of correlation functions

In this appendix, we explicitly calculate the limit

$$\lim_{\epsilon\to 0} E_K(a\epsilon,\, b\epsilon)\,, \tag{A.67}$$

and

$$\lim_{\epsilon\to 0} E_C(a\epsilon,\, b\epsilon,\, c\epsilon)\,. \tag{A.68}$$

We start from the expression (4.34). We take up the first term and consider the following limit: for $0 \le k \le 6$,

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_0^\infty ds\,\frac{s^k (a\epsilon)^{7-k}}{(s+a\epsilon+b\epsilon)^8}e^{-s}\cos\left(\frac{2\pi a\epsilon}{s+a\epsilon+b\epsilon}\right)\\
&=\lim_{\epsilon\to 0}\int_0^\infty ds\,\frac{s^k}{(s+\alpha)^8}e^{-\epsilon s}\cos\left(\frac{2\pi}{s+\alpha}\right)\\
&=\lim_{\epsilon\to 0}\sum_{l=0}^{\infty}\frac{(-)^l}{(2l)!}\int_0^\infty ds\,\frac{s^k}{(s+\alpha)^8}e^{-\epsilon s}\left(\frac{2\pi}{s+\alpha}\right)^{2l}\,,
\end{aligned} \tag{A.69}$$

where we put $\alpha = 1 + b/a$. For $k = 0$, the integral in this expression can be written as

$$\int_0^\infty ds\,\frac{1}{(s+\alpha)^8}e^{-\epsilon s}\left(\frac{1}{s+\alpha}\right)^{2l} = e^{\alpha\epsilon}\epsilon^{7+2l}\Gamma(-7-2l,\,\alpha\epsilon)\,, \tag{A.70}$$

where $\Gamma(z,\,\epsilon)$ denotes the incomplete gamma function defined by

$$\Gamma(z,\,\epsilon) \equiv \int_{\epsilon}^{\infty} e^{-t} t^{z-1} dt. \tag{A.71}$$

Using relations,

$$\lim_{\epsilon\to 0} \epsilon^k \Gamma(-k,\epsilon) = \frac{1}{k} \quad (k \in \mathbb{N})\,, \tag{A.72}$$

and

$$\frac{d^m}{d\epsilon^m}\left[\epsilon^k \Gamma(-k,\epsilon)\right] = (-1)^m \epsilon^{k-m} \Gamma(-k+m,\,\epsilon)\,, \tag{A.73}$$

we obtain that

$$\begin{aligned}
&\lim_{\epsilon\to 0} \int_0^\infty ds\, \frac{s^k (a\epsilon)^{7-k}}{(s+a\epsilon+b\epsilon)^8} e^{-s} \cos\left(\frac{2\pi a\epsilon}{s+a\epsilon+b\epsilon}\right) \\
=&(-1)^k \sum_{l=0}^{\infty} \frac{(2\pi i)^{2l}}{(2l)!} \alpha^{k-2l-7} \sum_{j=0}^{k} \left(\begin{array}{c} k \\ j \end{array}\right) \frac{(-1)^j}{7+2l-j} \\
=&k! \sum_{l=0}^{\infty} \frac{(2\pi i)^{2l}}{(2l)!} \frac{(2l+6-k)!}{(2l+7)!} \alpha^{k-2l-7}\,.
\end{aligned} \tag{A.74}$$

Note that, using some symbolic computation software, we can render this series into an expression with trigonometric functions.

Similarly, we obtain the following expressions; for $0 \le k \le 6$,

$$\begin{aligned}
&\lim_{\epsilon\to 0} \int_0^\infty ds\, \frac{s^k (a\epsilon)^{7-k}}{(s+a\epsilon+b\epsilon)^8} e^{-s} \sin\left(\frac{2\pi a\epsilon}{s+a\epsilon+b\epsilon}\right) \\
=&(-1)^k \sum_{l=0}^{\infty} \frac{2\pi(2\pi i)^{2l}}{(2l+1)!} \alpha^{k-2l-8} \sum_{j=0}^{k} \left(\begin{array}{c} k \\ j \end{array}\right) \frac{(-1)^j}{8+2l-j} \\
=&k! \sum_{l=0}^{\infty} \frac{2\pi(2\pi i)^{2l}}{(2l+1)!} \frac{(2l+7-k)!}{(2l+8)!} \alpha^{k-2l-8}\,,
\end{aligned} \tag{A.75}$$

$$\begin{aligned}
&\lim_{\epsilon\to 0} \int_0^\infty ds\, \frac{s^k ((a+b)\epsilon)^{7-k}}{(s+a\epsilon+b\epsilon)^8} e^{-s} \cos\left(\frac{2\pi(a\epsilon+b\epsilon)}{s+a\epsilon+b\epsilon}\right) \\
=&\sum_{l=0}^{\infty} \frac{(-1)^l}{(2l)!} \frac{(2l+k)!(6-k)!}{(2l+7)!} (2\pi)^{2l}\,,
\end{aligned} \tag{A.76}$$

$$\begin{aligned}
&\lim_{\epsilon\to 0} \int_0^\infty ds\, \frac{s^k ((a+b)\epsilon)^{7-k}}{(s+a\epsilon+b\epsilon)^8} e^{-s} \sin\left(\frac{2\pi(a\epsilon+b\epsilon)}{s+a\epsilon+b\epsilon}\right) \\
=&\sum_{l=0}^{\infty} \frac{(-1)^{l+1}}{(2l+1)!} \frac{(2l+k+1)!(6-k)!}{(2l+8)!} (2\pi)^{2l+1}\,.
\end{aligned} \tag{A.77}$$

Using above fomulae, we obtain the following expressions:

$$\lim_{\epsilon\to 0} E_K(a\epsilon,\, b\epsilon) = \frac{2}{\pi^2} + \frac{(a+b)^2 + 2\pi^2 ab}{\pi^2(a+b)^2}\cos\left(\frac{2a\pi}{a+b}\right) + \frac{a-b}{\pi(a+b)}\sin\left(\frac{2a\pi}{a+b}\right), \tag{A.78}$$

$$\begin{aligned}
&\lim_{\epsilon\to 0} E_C(a\epsilon,\, b\epsilon,\, c\epsilon) \\
=&-\frac{3}{\pi^2} - \frac{(2a-b-c)(a+b+c)^2 + 2\pi^2 a^2(b+c)}{\pi^2(a+b+c)^3}\cos\left(\frac{2\pi a}{a+b+c}\right) \\
&-\frac{(2b-c-a)(a+b+c)^2 + 2\pi^2 b^2(c+a)}{\pi^2(a+b+c)^3}\cos\left(\frac{2\pi b}{a+b+c}\right) \\
&-\frac{(2c-a-b)(a+b+c)^2 + 2\pi^2 c^2(a+b)}{\pi^2(a+b+c)^3}\cos\left(\frac{2\pi c}{a+b+c}\right) \\
&+\frac{3(a+b+c)^2 - 2\pi^2 a(a-2b-2c)}{2\pi^3(a+b+c)^2}\sin\left(\frac{2\pi a}{a+b+c}\right) \\
&+\frac{3(a+b+c)^2 - 2\pi^2 b(b-2c-2a)}{2\pi^3(a+b+c)^2}\sin\left(\frac{2\pi b}{a+b+c}\right) \\
&+\frac{3(a+b+c)^2 - 2\pi^2 c(c-2a-2b)}{2\pi^3(a+b+c)^2}\sin\left(\frac{2\pi c}{a+b+c}\right).
\end{aligned} \tag{A.79}$$

From these expressions, we see that $E_K(x,\, y)$ and $E_C(x,\, y,\, z)$ satisfy conditions corresponding to (3.15) and (3.16), and accordingly, we can use the formulae (4.33) and (4.43).

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